R-manifolds and multivalued solutions of PDE's

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- Jet manifolds and PDE.
  - Introduction: jet spaces $J^k(n, m)$.
  - Jets of vector bundles $J^k(\pi)$.
  - Jets of submanifolds $J^k(E, n)$.
  - Jets of fibring. Jets of mappings $M \to N$. Jets of functions.
  - Definitions of differential equation and its solution.
- Geometric interpretation of solutions. R-manifolds.
  - The Cartan distribution on $J^k(E, n)$.
  - Coordinate description of the Cartan distribution.
  - Geometric structure of the Cartan planes.
  - Description of integral submanifolds of the Cartan distribution.
  - Ray submanifolds, prolongations of integral submanifolds.
  - The structure of locally maximal integral submanifolds of the Cartan distribution.
  - R-manifolds.
  - Definitions of differential equation and its solution.
- Lie transformations (high order contact transformations).
  - Lie transformations. Point transformations.
  - Prolongations of point transformations. Prolongations of Lie transformations.
  - Prolongations of morphisms of contact structures.
  - Lie-Bäcklund theorem.

- Lie fields, liftings (prolongations) of Lie fields.
- Infinitesimal Lie-Bäcklund theorem.

- Extrinsic and intrinsic geometries of PDE.
  - External and internal points of view on PDE.
  - Problem of the reconstruction of the embedding $\mathcal{E} \to J^k$ and the Cartan distribution on J^k.
  - Extrinsic and intrinsic symmetries of PDE.
  - Rigidity. Examples.

Problems

1. Let $\pi : E^{n+m} \to M^n$ be a smooth bundle. Prove that $J^k(\pi)$ is an open everywhere dense subset of $J^k(E, n)$.

2. Let $\mathcal{E} \subset J^2(\mathbb{R}^3, 2)$ be the minimal surface equation. Prove that $\pi_{2,1} : \mathcal{E} \to J^1(\mathbb{R}^3, 2)$ is a nontrivial 2-dimensional vector bundle.

3. Let $k \geq l$. Prove that $J^k(E, n)$ is the manifold of (k-l)-jets of submanifolds of the form $L^{(l)}$ in $J^l(E, n)$.

4. Prove that the Cartan distribution on $J^k(E, n)$ is locally determined by the set of the Cartan forms

$$\omega^j_\sigma = dp^j_\sigma - \sum_{i=1}^{n} dp^j_{\sigma+1_i} dx_i, \ \mid \sigma \mid < k, \ j = 1, \ldots, m.$$

5. Let $\pi : E \to M$ be a fiber bundle and let $\pi_{1,0} : J^1(\pi) \to E$. Show that sections of the bundle $\pi_{1,0}$ are connections in the bundle π, while the condition of zero curvature determines a first order equation in the bundle $\pi_{1,0}$.

6. Consider the system of equations:

$$\left\{ \begin{array}{lcl} u_x & = & f(x, y, u) \\ u_y & = & g(x, y, u). \end{array} \right.$$

   Prove that if this system is compatible, then the Cartan distribution restricted to the corresponding surface is completely integrable.

7. Prove the following statements:

   (a) $L^{(k)}$ is a locally maximal integral manifold of the Cartan distribution.

   (b) Let $Q \subset J^k(E,n)$ be an n-dimensional integral manifold that is transversal to the fibers of the projection $\pi_{k,k-1}$. Then, locally, Q is of the form $L^{(k)}$.

8. (a) Let $\xi = X(x,y,z)\frac{\partial}{\partial x} + Y(x,y,z)\frac{\partial}{\partial y} + Z(x,y,z)\frac{\partial}{\partial z}$ be a nonzero vector at a point $\theta \in J^0(2,1)$, and P the straight line determined by this vector. Deduce equations describing the ray manifold $l(P)$.

   (b) Let W be a curve $x = \alpha(t), y = \beta(t), z = \gamma(t)$. Describe $\mathcal{L}(W)$.

   (c) Let W be a surface of the form $z = f(x,y)$. Describe $\mathcal{L}(W)$.

9. Prove that $\dim \mathcal{L}(W) = r + m\binom{k+n-r-1}{n-r-1}$, where $r = \dim W$.

10. Let $F : J^0(\pi) \to J^0(\pi)$ be a point transformation. Prove that if the lifting $F^{(1)}$ is defined in a neighborhood of a point $\theta \in J^1(\pi)$, then the lifting $F^{(k)}$ is defined in a neighborhood of any point θ' such that $\pi_{k,1}(\theta') = \theta$.

11. Prove that ordinary differential equations are non rigid.

Exercises

1. Prove that the family of neighborhoods $\pi_k^{-1}(U)$ together with the coordinate functions (x, p_σ^j) determines a smooth manifold structure in $J^k(\pi)$. Prove that $\dim J^k(\pi) = n + m\binom{n}{n+k}$

2. Prove that $\pi_k : J^k(\pi) \to M$ is a smooth locally trivial vector bundle.

3. Prove that $\pi_{k+1,k} : J^{k+1}(\pi) \to J^k(\pi)$ is a smooth locally trivial bundle (not vector bundle).

4. Prove that

   (a) $\pi_{l,s} \circ \pi_{k,l} = \pi_{k,s}, \;\; k \geq l \geq s$;

   (b) $\pi_l \circ \pi_{k,l} = \pi_k, \;\; k \geq l$;

(c) $\pi_{k,l} \circ j_k(s) = j_l(s),\ k \geq l$.

5. Prove that

   (a) $J^k(E, n)$ is a smooth manifold of dimension $n + m\binom{n}{n+k}$;

   (b) $\pi_{k,l} : J^k(E, n) \to J^l(E, n),\ k \geq l$ is a smooth locally trivial bundle;

   (c) $\pi_{l,s} \circ \pi_{k,l} = \pi_{k,s},\ \pi_{k,l} \circ j_k(L) = j_l(L),\ k \geq l \geq s$.

6. Let $F : J^k(E, n) \to J^k(E, n)$ be a Lie transformation. Prove that

   (a) $\pi_{k+l,k+s} \circ F^{(l)} = F^{(s)} \circ \pi_{k+s,k+l},\ l \geq s$;

   (b) $\mathrm{id}^{(s)} = \mathrm{id}$;

   (c) $(F \circ G)^{(s)} = F^{(s)} \circ G^{(s)}$.

7. Consider the Legendre transformation F in the space $J^1(2, 1)$: $\bar{x} = -p$, $\bar{y} = -q$, $\bar{u} = u - xp - yq$, $\bar{p} = x$, $\bar{q} = y$.

   (a) Prove that F is a Lie transformation;

   (b) Describe the lifting $F^{(1)}$;

   (c) Prove that F can not be represented in the form $F = G^{(1)}$, where G is a point transformation.

8. Let $F : E \to E'$ be a morphism over a diffeomorphism $\bar{F} : M \to M$ of two bundles π and π' over M. Define the prolongation $F^{(k)} : J^k(\pi) \to J^k(\pi')$ in such a way that $F^{(k)}$ will be a morphism of π_k in π'_k over $\bar{F}$.

9. Let $F : J^k(E, n) \to J^l(E, n)$, $k > l$ be a smooth surjection such that $F_* \mathcal{C}^{(k)}_\theta = \mathcal{C}^{(l)}_{F(\theta)}$. Define the prolongation $F^{(k)} : J^k(E, n) \to J^k(E, n)$.

10. For the vector field $X = x\frac{\partial}{\partial u} - u\frac{\partial}{\partial x}$ on $J^0(2, 1)$ find $X^{(2)}$.